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STURM-LIOUVILLE PROBLEMS WITH SEVERAL PARAMETERS. (U)
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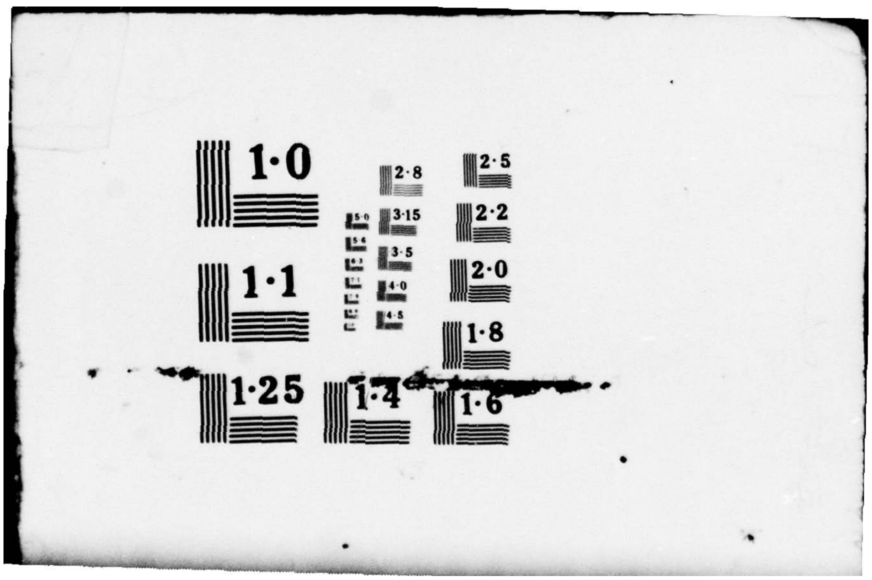
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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER (18) AFOSR-TR- 79-0921 (11)	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) STURM-LIOUVILLE PROBLEMS WITH SEVERAL PARAMETERS, (2)		5. TYPE OF REPORT & PERIOD COVERED (9) Interim rept.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) Lawrence Turyn		8. CONTRACT OR GRANT NUMBER(s) (15) AFOSR-76-3092
9. PERFORMING ORGANIZATION NAME AND ADDRESS Brown University Division of Applied Mathematics Providence, Rhode Island 02912		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F (16) 2304/A1 (17) A1
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332 (11)		12. REPORT DATE June 1979
		13. NUMBER OF PAGES 38
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 39p.		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

065 300

LEVEL II

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We consider the regular linear Sturm-Liouville problem (second-order linear ordinary differential equation with boundary conditions at two points $x = 0$ and $x = 1$, those conditions being separated & homogeneous) with several real parameters $\lambda_1, \ldots, \lambda_N$. Solutions to those problems correspond to eigenvalues $\underset{\sim}{\lambda} = (\lambda_1, \ldots, \lambda_N)$ lying on surfaces in $\mathbb{R}^N$ determined by the number of zeroes in $(0,1)$ of solutions. We describe properties of these surfaces, including: boundedness, and when unbounded,

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asymptotic directions. Using these properties some results are given for the system of N Sturm-Liouville problems which share only the parameters . Sharp results are given for the system of two problems sharing two parameters.

The eigensurfaces for a single problem are closely related to the cone $K = \{\lambda \in \mathbb{R}^N\colon \lambda_1 a_1(x) + \ldots + \lambda_N a_N(x) \leq 0$ for all x in $[0,1]\}$, particularly in questions of boundedness. The cone K and related objects are discussed, and a result is given which relates cones with two oscillation conditions known as "Right-Definiteness" and 'Left-Definiteness".

AFOSR-TR- 79-0921

STURM-LIOUVILLE PROBLEMS WITH SEVERAL PARAMETERS+

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+This research was supported in part by the Air Force Office of Scientific Research AFOSR 76-3092C.

STURM-LIOUVILLE PROBLEMS WITH SEVERAL PARAMETERS

LAWRENCE TURYN

ABSTRACT

We consider the regular linear Sturm-Liouville problem (second-order linear ordinary differential equation with boundary conditions at two points $x = 0$ and $x = 1$, those conditions being separated and homogeneous) with several real parameters $\lambda_1, \ldots, \lambda_N$. Solutions to this problem correspond to <u>eigenvalues</u> $\lambda = (\lambda_1, \ldots, \lambda_N)$ lying on surfaces in $\mathbb{R}^N$ determined by the number of zeroes in $(0,1)$ of solutions. We describe properties of these surfaces, including: boundedness, and when unbounded, asymptotic directions. Using these properties some results are given for the system of N Sturm-Liouville problems which share only the parameters λ. Sharp results are given for the system of two problems sharing two parameters.

The eigensurfaces for a single problem are closely related to the cone $K = \{\lambda \in \mathbb{R}^N: \lambda_1 a_1(x) + \ldots + \lambda_N a_N(x) \leq 0$ for all x in $[0,1]\}$, particularly in questions of boundedness. The cone K and related objects are discussed, and a result is given which relates cones with two oscillation conditions known as "Right-Definiteness" and "Left-Definiteness".

Section 1: The regular Sturm-Liouville problem with separated, homogeneous boundary conditions has been thoroughly studied. Recent authors have considered the generalization of this problem in which there are eigenvalue parameters $\lambda_1,\ldots,\lambda_N$ instead of a single parameter. This problem consists of the equation

$$(py')' + (\lambda_1 a_1(x) + \ldots \lambda_N a_N(x) + q(x))y = 0, \tag{1.1.N}$$

$x \in [0, 1]$, $' = \frac{d}{dx}$, along with the boundary conditions

$$\begin{cases} \cos\alpha \cdot y(0) - \sin\alpha \cdot p(0)y'(0) = 0 \\ \cos\beta \cdot y(1) - \sin\beta \cdot p(1)y'(1) = 0. \end{cases} \tag{1.2}$$

To make the problem regular we assume that $p, q, a_1, \ldots, a_N$ are continuous and real-valued and that p is positive and continuously differentiable. To make the eigenvalue problem non-trivial we will always assume that at least one of the a_j's is not identically zero. Sometimes we will call the a_j's the coefficients.

We will discuss the problem (1.1.N), (1.2), and then we will apply our results to a system of regular Sturm-Liouville problems which share the same parameters $\lambda_1,\ldots,\lambda_N$. The results for the system are part of Multiparameter Oscillation Theory.

For a non-trivial solution $y(x;\lambda)$ of (1.1.N) we define the oscillation number as the number of zeroes in $(0,1)$ of the function $y(x;\lambda)$. We define sets $S_n = \{\lambda^0 \in \mathbb{R}^N$: there is a non-trivial solution $y(x;\lambda^0)$ of (1.1.N), (1.2) having oscillation number $n\}$. If $\lambda \in S_n$ for some n we call λ an <u>eigenvalue</u>. A number n^* such that $S_n = \phi$ for $n < n^*$ and $S_n \neq \phi$ for $n \geq n^*$ is called the <u>minimum oscillation number</u>. It will be proven in Theorem 2.3 that n^* always exists.

Define the ray through a point $0 \neq \lambda^0 \in \mathbb{R}^N$ as the set $\mathscr{R}(\lambda^0) = \{t\lambda^0: t \geq 0\}$. We will investigate the problem (1.1.N), (1.2) in part by looking along rays; this amounts to writing $\mathbb{R}^N \approx$ (the unit sphere) $\times [0,\infty)$.

As a note on terminology, whenever a function is called analytic we mean real-analytic.

Given the functions $a_1,\ldots,a_N$, notate $\lambda\cdot a(x) = \lambda_1 a_1(x) + \ldots + \lambda_N a_N(x)$ and define a cone $K = \{\lambda \in \mathbb{R}^N: \lambda\cdot a \leq 0$ on $[0,1]\}$. It will be shown in Section 3 that the eigensurfaces are closely related to this cone: Let S_n be any non-empty eigensurface; then S_n is bounded if and only if $K = \{0\}$. Let us suppose now that $K \neq \{0\}$, so that S_n is unbounded. Let $t_m\lambda^m \in S_n$ with $|\lambda^m| = 1$ and $t_m \to \infty$. Since the unit sphere is compact, there is a λ^0 and a convergent subsequence $\lambda^{m'} \to \lambda^0$. Such a λ^0 will be called an <u>asymptote</u>, and we will say that S_n <u>has asymptotics</u> $\lambda^{m'} \to \lambda^0$. In Theorems 3.2, 3.3 it will be shown that $0 \neq \lambda^0$ is an asymptote if and only if $\lambda^0 \in \partial \overset{\text{defn}}{=} \partial K$. For the two-parameter problem ($N = 2$) it will be shown in Theorem 3.4 that the eigencurves are asymptotically parallel to ∂, whenever $K \neq \{0\}$.

In Section 4, we will consider the system

$$(p_i y_i'(x_i))' + (\lambda_1 q_{i1}(x_i) + \dots + \lambda_N a_{iN}(x_i) + q_i(x_i))y_i = 0$$

$$\cos \alpha_i \cdot y_i(0) - \sin \alpha_i \cdot p_i(0) y_i'(0) = 0$$

$$\cos \beta_i \cdot y_i(1) - \sin \beta_i \cdot p_i(1) y_i'(1) = 0,$$

$i = 1,\dots,N$, of regular Sturm-Liouville problems. For each of the problems there will be surfaces S_n^i. Notate $\lambda \cdot a_{i*}(x_i) = \lambda_1 a_{i1}(x_i) + \dots + \lambda_N a_{iN}(x_i)$ and define cones $K_i = \{\lambda \in \mathbb{R}^N: \lambda \cdot a_{i*} \leq 0$ on $[0,1]\}$, $C_i^- = \{\lambda \in \mathbb{R}^N: \lambda \cdot a_{i*} < 0$ on $[0,1]\}$, and $\partial_i = \partial K_i$. In Theorem 4.2 it will be shown that when the surfaces S_n^i intersect nicely, for example if there are integers N_i^* such that $\bigcap_{i=1}^N S_{n_i}^i \neq \phi$ whenever $n_i \geq N_i^*$, $i = 1,\dots,N$, necessarily the cones intersect nicely: $(\bigcap_{j \neq i} \partial_j) \cap (\mathbb{R}^N \setminus \text{int } K_i) \neq \phi$, $i =$ $i = 1,\dots,N$. For the two-parameter problem, Theorem 4.5 proves that $S_{n_1}^1 \cap S_{n_2}^2 \neq \phi$ whenever $n_1 \geq 0$, $n_2 \geq 0$, as long as some conditions on the cones are satisfied: $\partial_1 \cap \text{ext } K_2 \neq \phi$, $\text{ext } K_1 \cap \partial_2 \neq \phi$, and a technical assumption. It turns out that Richardson's result, Theorem 4.6, has as its corollaries all other results for the two-parameter problem which have appeared in the literature. An oscillation result is proven as Theorem 4.8, and an example is given which shows that the result is new.

In Section 5 the cones are discussed. Theorem 5.1 states that either $C^- = \phi$ or $C^- = \text{int } K$, and this seems to be a new result. In Theorem 5.2 a linear algebra result is given: Assuming that $\bigcap_{i=1}^N C_i^- \neq \phi$, the condition known as "Left-Definiteness" of Källström and Sleeman [13] implies the condition known as "Right-Definiteness".

Section 2: For the Sturm-Liouville problem consisting of an equation

$$(2.1) \qquad (py')' + gy = 0$$

along with boundary conditions

$$(2.2) \qquad \cos\alpha\cdot y(0) - \sin\alpha\cdot p(0)y'(0) = 0$$

$$(2.3) \qquad \cos\beta\cdot y(1) - \sin\beta\cdot p(1)y'(1) = 0$$

it is advantageous to make the Prüfer substitution $y(x) = R(x)\sin\theta(x)$, $p(x)y'(x) = R(x)\cos\theta(x)$. Problems (2.1)-(2.3) is then equivalent (see Coddington and Levinson [2, pp. 208-213]) to the problem consisting of the equations

$$(2.4) \qquad R'(x) = (p(x)^{-1} - g(x))R(x)\sin\theta(x)\cos\theta(x)$$

$$(2.5) \qquad \theta'(x) = p(x)^{-1}\cos^2\theta(x) + g(x)\sin^2\theta(x)$$

along with the initial condition

$$(2.6) \qquad \theta(0) = \alpha$$

and the end condition

$$(2.7) \qquad \theta(1) \equiv \beta(\mathrm{mod}\ \pi).$$

Without loss of generality $0 \leq \alpha < \pi$ and $0 \leq \beta \leq \pi$; it will be shown below that without loss of generality $\beta > 0$.

At any x where $\theta(x) \equiv 0 (\mathrm{mod}\ \pi)$ the corresponding solution $y(x)$ has a zero, and vice-versa. Whenever $\theta(x) \equiv 0 (\mathrm{mod}\ \pi)$ necessarily $\theta'(x) = p(x)^{-1} > 0$, so the graph of the function $\theta(x)$ versus x can cross the lines $\theta \equiv 0 (\mathrm{mod}\ \pi)$ only in the upward direction. From this it follows that if $\theta(x_0) \geq k\pi$ for some integer $k \geq 0$ then necessarily $\theta(x) > k\pi$ for all $x > x_0$. In particular, $\theta(1) > 0$, so that without loss of generality $\beta > 0$. We also note that $n\pi < \theta(1) \leq (n+1)\pi$ if and only if the corresponding solution has oscillation number n.

The most basic result in Sturm Comparison Theory compares the solutions of two initial value problems:

Theorem 2.1 (Coddington and Levinson [2, p. 210]): Suppose that there are two initial value problems of the form (2.5), (2.6), that is, choices p_i, g_i, α_i and corresponding solutions θ_i, $i = 1,2$. If $\alpha_2 \geq \alpha_1$, $p_2 \leq p_1$, and $g_2 \geq g_1$, then $\theta_2(x) \geq \theta_1(x)$ for all $x \geq 0$. If further $g_2 > g_1$ except possibly at isolated points, or if $\alpha_2 > \alpha_1$, then $\theta_2(x) > \theta_1(x)$ for all $x > 0$.

From this result we can conclude that the solution of the problem which corresponds to (p_2, q_2, α_2) has an oscillation number at least as great as that of the solution of the problem which corresponds to (p_1, q_1, α_1).

Under a strong hypothesis it is possible to prove the existence of eigenvalues for the one-parameter problem which consists of the equation

$$(1.1.1) \qquad (py')' + (\lambda_1 a_1 + q)y = 0$$

along with boundary conditions (1.2).

Theorem 2.2 (Coddington and Levinson [2, p. 212]): Assume that $a_1 > 0$ on $[0,1]$. Then, for every integer $n \geq 0$, there is a unique eigenvalue λ_1^n with the corresponding solution to (1.1.1), (1.2) having exactly n oscillations. Further, $\theta(1;\lambda_1) \to 0$ as $\lambda_1 \to -\infty$.

When $g(x) = g(x,\lambda) = \lambda \cdot a(x) + q(x)$, $\lambda \in \mathbb{R}^N$, we will denote the corresponding solution of (2.5), (2.6) by $\theta(x) = \theta(x;\lambda)$. Theorem 2.2 is equivalent to saying that when $N = 1$, $a_1 > 0$, there is a unique eigenvalue λ_1^n with $\theta(1;\lambda_1^n) = n\pi + \beta$.

For arbitrary N it is known that $\theta(1;\lambda)$ is an analytic function of λ; see Hale [1, pp. 21-22]. From this we have

Theorem 2.3: The minimum oscillation number always exists.

Proof: Define the function $\psi: \mathbb{R}^N \to (0,\infty): \lambda \mapsto \theta(1;\lambda)$. Since $\mathbb{R}^N$ is connected, $\psi(\mathbb{R}^N)$ is connected and hence either a single point or an interval, possibly infinite to the right. Since we assume that at least one of the a_j is non-trivial we can show that $\psi(\mathbb{R}^N)$ is an interval which is infinite to the right: For if $a_1(x_0)$ is positive (exactly the same sort of argument will apply if $a_1(x_0) < 0$) say, then $a_1(x)$ is positive on some small subinterval about x_0. Thus, $\theta(x_0;\lambda) \to \infty$ as $\lambda_1 \to \infty$ when

$\lambda_2, \ldots, \lambda_N$ are held fixed. It follows then that $\psi(\lambda) \to +\infty$ as $\lambda_1 \to \infty$ when $\lambda_2, \ldots, \lambda_N$ are held fixed (recall the discussion on page 5). One then defines $n^* = \min\{n \geq 0: n\pi + \beta \in \psi(\mathbb{R}^N)\}$. ∎

The results in Theorem 2.3 has been proven for the one-parameter problem by Eisenfeld [3] and Faierman [6] using more explicit arguments. Those arguments yield more information about the eigenvalues for the one-parameter problem, for example that there is a sequence $\lambda^n \in S_n$ that is monotonic, i.e. $\lambda^n < \lambda^{n+1}$ for $n \geq n^*$, or $\lambda^n < \lambda^{n+1}$ for $n \geq n^*$.

To close this section we will discuss the convexity of the eigensurfaces. A region $R \subset \mathbb{R}^N$ is called <u>convex</u> if $t\lambda + (1-t)\mu \in R$ whenever $\lambda, \mu \in R$ and $0 < t < 1$. A surface S which is the boundary of a region R will be called <u>convex</u> if R is convex. Let us define now regions $R_n = \{\lambda \in \mathbb{R}^N: \theta(1;\lambda) < n\pi + \beta\}$, and note that S_n is the boundary of R_n. The following example shows that the eigensurfaces S_n are not necessarily convex:

<u>Example 2.4</u>: Let $q = 0$, $\alpha = 0$, $\beta = \pi$, and consider the two-parameter problem

$$y''(x) + \lambda \cdot a(x)y(x) = 0, \quad x \in [0,1]$$

$$y(0) = y(1) = 0$$

where $a_1(x) = 1$ for $0 \leq x \leq \frac{1}{2}$, $a_1(x) = 0$ for $\frac{1}{2} < x \leq 1$ and $a_2(x) = 0$ for $0 \leq x \leq \frac{1}{2}$, $a_2(x) = 1$ for $\frac{1}{2} < x \leq 1$. For convenience in the calculations the coefficients a_j chosen are not continuous. The result of non-convexity will not be disturbed by the discontinuity of these a_j because these a_j can be approximated arbitrarily well in $L_1(0,1)$ by continuous functions, and the angle function θ depends continuously on the a_j in $L_1(0,1)$.

Take $\lambda^1 = (8\pi^2, 0)$, $\lambda^2 = (0, 8\pi^2)$, and $\lambda^3 = (\lambda^1 + \lambda^2)/2 = = (4\pi^2, 4\pi^2) =$ (a convex combination). We will show that $\lambda^1, \lambda^2 \in \text{int } R_2$ and that $\lambda^3 \in S_2 = \partial R_2$. We calculate the solution $y(x;\lambda^3) = (2\pi)^{-1}\sin(2\pi x)$, so that $\lambda^3 \in S_2$. We also calculate the solution

$$y(x;\lambda^1) = \begin{cases} \omega^{-1}\sin(\omega x) & , \ 0 \leq x \leq \frac{1}{2} \\ \omega^{-1}\sin(\omega/2) + \cos(\omega/2)\cdot(x - \frac{1}{2}), & \frac{1}{2} < x \leq 1 \end{cases}$$

where $\omega = 2^{3/2}\pi$, from which we conclude that $y(x;\lambda^1)$ has exactly one zero on $(0,\frac{1}{2}]$ and does not vanish on $(\frac{1}{2},1]$. So $\lambda^1 \in \text{int } R_2$. One can conclude the same for λ^2.

Section 3: For the N-parameter problem (1.1.N), (1.2) the eigensurfaces S_n are closely related to the cone $K = \{\lambda \in \mathbb{R}^N: \lambda \cdot a(x) \leq 0 \ \forall x \in [0,1]\}$. Recall that λ^0, $|\lambda^0| = 1$, is an asymptote for S_n if there exists a sequence $t_m\lambda^m \in S_n$, $|\lambda^m| = 1$, $t_m \to \infty$, with $\lambda^m \to \lambda^0$. Let S_n be any non-empty eigensurface. Then (i) S_n is bounded if and only if $K = \{0\}$, and (ii) $0 \neq \lambda^0$ is an asymptote for S_n if and only if $\lambda^0 \in \partial = \partial K$.

Part (i) will be proven in Theorems 3.1, 3.3, and part (ii) will be proven in Theorems 3.2, 3.3.

Define $\tau(\lambda) = \max\limits_{0 \leq x \leq 1} \tau \cdot a(x)$. *Note that* $\tau(\cdot)$ *is continuous*, because $\lambda^m \to \lambda^0$ implies $\lambda^m \cdot a \to \lambda^0 \cdot a$ uniformly on $[0,1]$, and note also that $\lambda \in K$ if and only if $\tau(\lambda) \leq 0$.

Theorem 3.1: $K = \{0\}$ implies that every non-empty S_n is bounded.

Proof: $K = \{0\}$ implies $\tau(\lambda) > 0$ for all $\lambda \neq 0$, in particular for all $|\lambda| = 1$. Since the unit sphere is compact and $\tau(\cdot)$ is continuous, there is a $\tau_0 > 0$ such that $\tau(\lambda) \geq \tau_0$ for all $|\lambda| = 1$.

Suppose now that there is an $\bar{n}$ with $S_{\bar{n}}$ unbounded. Let $t_m\lambda^m \in S_{\bar{n}}$ with $t_m \to \infty$ and $|\lambda^m| = 1$. By taking a subsequence, we may assume that $\lambda^m \to \lambda^0$ for some $|\lambda^0| = 1$. Since $\lambda^m \cdot a \to \lambda^0 \cdot a$ uniformly on $[0,1]$, there is an interval I_0, of non-zero length, and an integer M such that $\lambda^m \cdot a(x) \geq \tau_0/2$ for all $x \in I_0$ and for all $m \geq M$. If $x_0 \in \text{int } I_0$ then $\theta(x_0; t_m\lambda^m) \to \infty$

as $m \to \infty$. This implies that $\theta(1;t_m\lambda^m) \to \infty$ as $m \to \infty$ (recall the discussion on page 5). This gives a contradiction. ■

Theorem 3.2: If λ^0 is an asymptote then $\lambda^0 \in \partial$.

Proof: For some $n \geq n^*$ there is a sequence $t_m\lambda^m \in S_n$ with $|\lambda^m| = 1$, $t_m \to +\infty$, and $\lambda^m \to \lambda^0$, by the definition of asymptotics. First we will show $\lambda^0 \in K$, and then we will show that $\lambda^0 \notin \text{int } K$.

To show that $\lambda^0 \in K$, it will suffice to show that $\tau(\lambda^m) \to 0$ as $m \to \infty$. Assume to the contrary that there is a $\delta > 0$ and subsequence $m' \to \infty$ with $\tau(\lambda^{m'}) \geq \delta$. Just as in the proof of Theorem 3.1 we can conclude that $\theta(1;\, t_{m'}\lambda^{m'}) \to \infty$ as $m' \to \infty$, giving a contradiction.

Next, we will show that $\lambda^0 \notin \text{int } K$: Let $\mu^0 \in S_{n*}$ so that $\theta(1;\mu^0) = n^*\pi + \beta$. Translate the parameters by μ^0 and consider the equation $(py')' + (\tilde{\lambda}\cdot a + (\mu^0\cdot a+q))y = 0$ along with boundary conditions (1.2). Let $\tilde{\theta}(x;\lambda)$ be the corresponding angle function for this problem, and note that it has eigenvalues $\tilde{\lambda}_m = t_m\lambda^m - \mu^0 = t_m(\lambda_m - t_m^{-1}\mu^0)$. Define the sequence $\mu^m = \lambda^m - t^{-1}\mu^0$, and note that it has the same asymptote λ^0 as does the sequence λ^m. In other words, a translation does not affect the asymptotics.

So let us suppose that $\lambda^0 \in \text{int } K$. We note that $\tilde{\lambda} \in \text{int } K$ implies that $\tilde{\lambda}\cdot a \leq 0$ with inequality somewhere on $[0,1]$, and this implies that $\tilde{\theta}(1;\tilde{\lambda}) < \tilde{\theta}(1;0)$ by comparison theory. (To see this, use Theorem 2.1 (Comparisons) on a small subinterval

(x_0, x_1) on which $\tilde{\lambda}\cdot a(x) < 0$ so that $\tilde{\theta}(x_1;\tilde{\lambda}) < \tilde{\theta}(x_1;0)$. Then use Theorem 2.1 on the subinterval $(x_1, 1]$ with $\tilde{\theta}(\cdot;\tilde{\lambda})$ having smaller initial data then $\tilde{\theta}(\cdot;0)$.) Since $\lambda^0 \in \operatorname{int} K$ and $\mu^m \to \lambda^0$, eventually $\mu^m \in \operatorname{int} K$, so that eventually

$$\theta(1;t_m\lambda^m) = \tilde{\theta}(1;t_m\mu^m) < \tilde{\theta}(1;0) = \theta(1;\mu^0) = n^*\pi + \beta \leq n\pi + \beta.$$

This gives a contradiction with $t_m\lambda^m \in S_n$. ■

Theorem 3.3: $K \neq \{0\}$ implies that every eigensurface is unbounded (hence has asymptotics) and that λ^0 is an asymptote whenever $|\lambda^0| = 1$ and $\lambda^0 \in \partial$.

Proof: Let n^* be the minimum oscillation number. First we will discuss S_n for $n > n^*$, and then S_{n^*} will be discussed in two distinct cases.

Recall that $\mathcal{R}(\lambda) = \{t\lambda : t \geq 0\}$. Since $\tau(\lambda) \leq 0$ if and only if $\lambda \in K$, for any fixed $\lambda \in \operatorname{ext} K$ there exists $n^*(\lambda) = \min\{n \geq 0 : S_n \cap \mathcal{R}(\lambda) \neq \phi\} < \infty$. As in the proof of Theorem 2.3, $\lambda \in \operatorname{ext} K$ implies $S_n \cap \mathcal{R}(\lambda) \neq \phi$ for all $n \geq n^*(\lambda)$, by using the distance along the ray as the only parameter.

Let us assume that the parameters have already been translated by a $\mu^0 \in S_{n^*}$, as was done in the proof of Theorem 3.2, so that $\theta(1;0) = n^*\pi + \beta$. This translation in no way affects the existence of asymptotics.

Choose any $\lambda^0 \in \partial$ for which $|\lambda^0| = 1$; this is possible because $K \neq \{0\}$. Choose any sequence $\lambda^m \to \lambda^0$ with $|\lambda^m| = 1$ and $\lambda^m \in \text{ext } K$; this is possible because K must be contained in a half-plane. We will show that for any fixed $n > n^*$ there is a sequence $t_m \geq 0$ with $t_m\lambda^m \in S_n$, after which it will be easy to show that necessarily $t_m \to +\infty$ so that λ^0 is an asymptote.

To show the existence of the t_m, it suffices to show that $n^*(\lambda^m) \leq (1+n^*)$ $\forall\, m$, because the set $\{\theta(1;t\lambda^m): t \geq 0\}$ is for each fixed m an interval infinite to the right, as in the proof of Theorem 2.3. Let $\psi(t) = \max\limits_m \theta(1;t\lambda^m)$, which is continuous for $t \geq 0$. Since $\psi(t) \to n^*\pi + \beta = \theta(1;0)$ as $t \searrow 0+$, there exists $t_0 > 0$.∋. $\theta(1;t_0\lambda^m) \leq (1+n^*)\pi$ $\forall\, m$.

Suppose now that we have fixed an $n > n^*$ and found the sequence $t_m \geq 0$ with $t_m\lambda^m \in S_n$, as was shown to be possible by the arguments above. If $t_m \to +\infty$ fails to hold then there is a bounded subsequence $\{t_{m'}\}$ which can be assumed to be convergent. Say that $t_{m'} \to \overline{t} \geq 0$. Then

$$\theta(1;t_{m'}\lambda^{m'}) \to \theta(1;\overline{t}\lambda^0) \leq \theta(1;0) = n^*\pi + \beta$$

by comparison theory and $\lambda^0 \in K$. This gives a contradiction with $t_m\lambda^m \in S_n$, $n > n^*$.

To show that the conclusions hold for $n = n^*$ as well as for $n > n^*$ we will consider two distinct cases: (i) there exists $\hat{\lambda}$ such that $\theta(1;\hat{\lambda}) < n^*\pi + \beta$, or (ii) there does not exist such a $\hat{\lambda}$. In case (i) we merely translate the parameters by $\hat{\lambda}$ instead of

μ^0 and the arguments above work, with only minor modification. In case (ii) necessarily there does not exist a $\overline{\lambda}$ with $\overline{\lambda}\cdot a \leq 0$ with strict inequality somewhere on $[0,1]$, for if there did exist such a $\overline{\lambda}$ then $\theta(1;\overline{\lambda}) < \theta(1;0) = n^*\pi + \beta$ by comparison theory. Because $\lambda \in \text{int } K$ implies $\lambda\cdot a \leq 0$ with strict inequality somewhere on $[0,1]$, $\text{int } K = \phi$ and $K = \partial = \{\lambda: \lambda\cdot a \equiv 0 \text{ on } [0,1]\}$. But then $(\mu^0+\lambda) \in S_{n^*}$ for all $\lambda \in \partial$! Thus the conclusions also hold in this case. ■

For the two-parameter problem we have eigencurves, and we can say more about the asymptotics since the possibilities for unbounded sequences of eigenvalues are, as one might expect, more limited than for $N \geq 3$.

We will say that an unbounded set $\Omega \subset \mathbb{R}^2$ is <u>asymptotically parallel to a ray</u> $\mathscr{R}(\lambda^0)$, $|\lambda^0| = 1$, if for all $\varepsilon > 0$ there is an $R > 0$ such that $t\lambda \in \Omega \cap \{|\mu| \geq R\}$, $|\lambda| = 1$, $t > 0$ implies that $|\lambda-\lambda^0| < \varepsilon$. Suppose now that we have a convex cone $C \subset \mathbb{R}^2$. It is not difficult to see that C is bounded by at most two rays, so that $\partial C = \mathscr{R}' \cup \mathscr{R}''$, with possibly $\mathscr{R}' = \mathscr{R}''$. We will say that a set $S \subset \mathbb{R}^2$ is <u>asymptotically parallel</u> to ∂C if there are (non-empty) unbounded sets S', S'' such that S is the disjoint union of S' and S'' and such that S' is asymptotically parallel to $\mathscr{R}'$ and S'' is asymptotically parallel to $\mathscr{R}''$.

In defining "Ω is asymptotically parallel to a ray..." we do not require that Ω is an arc; in fact, within the possibilities allowed by the definition we may have $\text{int } \Omega \neq \phi$ or $\Omega =$ (the disjoint union of two arcs).

In the λ-plane every non-trivial vector $\lambda = (\lambda_1, \lambda_2)$ has the representation $\lambda_1 = r \cdot \cos \gamma$, $\lambda_2 = r \cdot \sin \gamma$, with $r > 0$ and $0 \leq \delta < 2\pi$. With these restrictions the r, γ are uniquely defined, so we have a well-defined map $\gamma \colon \mathbb{R}^2 \to [0, 2\pi) \colon \lambda \mapsto \gamma(\lambda)$.

Theorem 3.4: For the two-parameter problem assume that $K \neq \{0\}$. Then every non-empty eigencurve S_n is asymptotically parallel to $\partial = \partial K$.

Proof: Let $\partial = \mathscr{R}^+ \cup \mathscr{R}^-$, and let $\lambda^{\pm}$ be such that $\mathscr{R}^{\pm} = \mathscr{R}(\lambda^{\pm}) = \{t\lambda^{\pm} \colon t \geq 0\}$, $|\lambda^{\pm}| = 1$. It is possible that $\mathscr{R}^+ = \mathscr{R}^-$. Let $\gamma^{\pm} = \gamma(\lambda^{\pm}) \in [0, 2\pi)$. Without loss of generality $\gamma^- \leq \gamma^+$. Let n^* denote the minimum oscillation number, as usual. Let $\mu^0 \in S_{n^*}$, so that $\theta(1; \mu^0) = (n^*\pi + \beta)$.

There are only two possibilities for the cone K, because it is convex: case (i) $K = \{t\lambda \colon t \geq 0,\ \gamma^- \leq \gamma(\lambda) \leq \gamma^+\}$, or case (ii) $K = \{t\lambda \colon t \in \mathbb{R},\ \gamma(\lambda) = \gamma^-\} = \partial = \{t\lambda \colon t \geq 0,\ \gamma(\lambda) = \pm\gamma^- (\mathrm{mod}\ 2\pi)\}$. Case (ii) can occur only when a_1, a_2 are linearly dependent. If a_1 is non-trivial, then case (ii) can occur only when a_1 is of both signs.

Case (ii) is simple and will be discussed first: Since a_1 is of both signs, there are eigenvalues $\mu_n^{\pm}$, $n \geq n^*$, for the one-parameter problem which consists of the equation $(py')' + (\mu a_1 + (\mu^0 \cdot a + q)\, y = 0$ along with the boundary conditions (1.2). In fact, from Theorem 2.3 (Minimum Oscillation Number) and the arguments contained in the proof of that theorem (see also Faierman [6]) we can conclude that for $n > n^*$ there exist positive integers $j^{\pm}(n)$ and eigenvalues $\mu^+_{n,k} > 0$, $k = 1,\ldots,j^+(n)$ and $\mu^-_{n,k} < 0$, $k = 1,\ldots,j^-(n)$ for this one-parameter problem. For the two-parameter problem, let a_1 be non-trivial and let $a_2 = \zeta\cdot a$ for some $\zeta \in \mathbb{R}$. Then the two-parameter problem which consists of the equation $(py')' + (\lambda_1 a_1+\lambda_2 a_2+q)y = 0$ along with boundary conditions (1.2) has "eigencurves"

$$S_n = (\textstyle\bigcup_{k=1}^{j^+(n)}\{\mu^0 + (\lambda_1,\lambda_2) : \lambda_1 + \zeta\cdot\lambda_2 = \mu^+_{n,k}\})$$

$$\cup\ (\textstyle\bigcup_{k=1}^{j^-(n)}\{\mu^0 + (\lambda_1,\lambda_2) : \lambda_1 + \zeta\cdot\lambda_2 = \mu^-_{n,k}\})$$

for $n > n^*$, i.e. a finite union of straight lines. Also, $\mu^0 + \partial = \{\mu^0 + \mu : \mu \in \partial\} \subset S_{n^*}$. In fact, all of these straight lines are parallel to ∂ = (a straight line), and the straight line $\mu^0 + \partial$ divides S_n into two groups of straight lines for $n > n^*$. Of course, this is an unusual case! If we define regions $R_- = \{t\lambda : t \geq 0,\ \gamma^- - (\pi/2) \leq \gamma(\lambda) \leq \gamma^- + (\pi/2),\ (\mathrm{mod}\ 2\pi)\}$, $R_+ = \mathbb{R}^2\setminus R_-$, and denote $S^{\pm} = S_n \cap R_{\pm}$, then we have a decomposition $S_n = S^+ \cup S^-$, as desired in the conclusions of this theorem.

Let us consider case (i) now. From the discussion within the proof of Theorem 3.3 we can conclude that there is a $\hat{\lambda}$ with $\theta(1;\hat{\lambda}) < (n^*\pi+\beta)$. Fix now any $n \geq n^*$. Let $\lambda^{m,\pm}$ be chosen such that $|\lambda^{m,\pm}| = 1$ and $\gamma(\lambda^{m,\pm}) = \gamma^{\pm}\cdot(1 \pm \frac{1}{2m})$. (Here we assume that $\gamma^- > 0$, which can be accomplished, if necessary when $\gamma^- = 0$, by a small rotation of $\mathbb{R}^2$ while leaving $\gamma^+ < 2\pi$.) Note that $\lambda^{m,\pm} \in \text{ext } K$. By the arguments in the proof of Theorem 3.3, there exist $t_{m,\pm} > 0$ such that $\mu^{m,\pm} \overset{\text{defn}}{=} (\hat{\lambda} + t_{m,\pm}\lambda^{m,\pm}) \in S_n$, $m \geq 1$, and necessarily $t_{m,\varepsilon} \to \infty$ as $m \to \infty$, $\varepsilon = +$ or $-$.

Define now $R_- = \{\hat{\lambda} + t\lambda: t \geq 0,\ 0 \leq \gamma(\lambda) < \gamma^-\}$, $R_+ = \{\hat{\lambda} + t\lambda: t \geq 0,\ \gamma^+ < \gamma(\lambda) < 2\pi\}$, and denote $S^\pm = S_n \cap R_\pm$. Because $\theta(1;\hat{\lambda}+\lambda) \leq \theta(1;\hat{\lambda}) < (n^*\pi+\beta)$ whenever $\gamma^- \leq \gamma(\lambda) \leq \gamma^+$, we have $S_n \cap \{t\lambda: t \geq 0,\ \gamma^- \leq \gamma(\lambda) \leq \gamma^+\} = \phi$. So $S_n = S^+ \cup S^-$, and S^ε is unbounded for $\varepsilon = +$ or $-$. If we can show that S^- is asymptotically parallel to $\mathscr{R}^-$, then the same arguments will show that S^+ is asymptotically parallel to $\mathscr{R}^+$, and the proof will be complete.

Suppose to the contrary that S^- is not asymptotically parallel to $\mathscr{R}^-$. Then there is a sequence $(\hat{\lambda}+t_m\lambda^m) \in S^-$ with $|\lambda^m| = 1$, $t_m \geq 0$, $t_m \to +\infty$, and $(\lambda^m+t_m^{-1}\hat{\lambda}) \not\to \lambda^-$. This implies that $\lambda^m \not\to \lambda^-$. Since the unit sphere is compact, there is a convergent subsequence, say $\lambda^{m'} \to \lambda^0$. By Theorem 3.2, either $\lambda^0 = \lambda^+$ or $\lambda^0 = \lambda^-$. By the definition of S^-, $\gamma(\lambda^m) < \gamma^-$. Because of this $\gamma(\lambda^0) \in [0,\gamma^-]$. This shows that $\gamma(\lambda^0) = \gamma^-$; since $|\lambda^0| = 1$, necessarily $\lambda^0 = \lambda^-$. This gives a contradiction, and completes the proof of the theorem. ■

Other work on asymptotics can be found in Faierman [7-9].

Section 4: Let us define for a given problem (1.1.N), (1.2) the minimum distance $\rho_n = \min_{\lambda \in S_n} |\lambda|$ for $n \geq n^*$. It is not difficult to show that the minimum distance is always achieved, i.e. there exists $\lambda^n \in S_n$ with $\rho_n = |\lambda^n|$, by using the representation $t\lambda$, $|\lambda| = 1$, for all elements of S_n. We have also

Lemma 4.1: $\rho_n \to \infty$ as $n \to \infty$.

Proof: If not, there is a subsequence $n' \to \infty$ and eigenvalues $\lambda^{n'} \in S_{n'}$ with $|\lambda^{n'}| = \rho_{n'}$ bounded. Since there are constants $k < \infty$, $p_0 > 0$ such that $k \geq |\lambda^{n'} \cdot a(x) + q(x)|$ for all n' and for all x in $[0,1]$ and $p_0 \leq p(x)$ for all x in $[0,1]$, we have $\theta(1;\lambda^{n'}) \leq (k/p_0)^{1/2} + \pi$ for all n', by Theorem 2.1 (Comparisons). This gives a contradiction with $n' \to \infty$. ∎

Suppose now that we are given a system of regular Sturm-Liouville problems of the form (1.1.N), (1.2), i.e. for $i = 1,\ldots,N$

$$
(4.1)\quad \begin{cases} (p_i y_i'(x_i))' + (\lambda_1 a_{i1}(x_i) + \ldots + \lambda_N a_{iN}(x_i) + q_i(x_i))y_i = 0 \\ \cos\alpha_i \cdot y_i(0) - \sin\alpha_i \cdot p_i(0) y_i'(0) = 0 \\ \cos\beta_i \cdot y_i(1) - \sin\beta_i \cdot p_i(1) y_i'(1) = 0. \end{cases}
$$

So that each of these problems is regular we assume that, for $i = 1,\ldots,N$, $p_i, q_i, a_{i1}, \ldots, a_{iN}$ are continuous and real-valued and that p_i is positive and continuously differentiable. Recall the notation $\lambda \cdot a_{i*}(x_i) = \lambda_1 a_{i1}(x_i) + \ldots + \lambda_N a_{iN}(x_i)$. Without loss of generality we assume that $0 \leq \alpha_i < \pi$ and $0 < \beta_i \leq \pi$. Corresponding to the i^{th} problem in the system is an angle function $\theta_i(x_i;\lambda)$, eigensurfaces S_n^i, minimum oscillation number n_i^*, minimum distances ρ_n^i, and cones $K_i = \{\lambda \in \mathbb{R}^N: \lambda \cdot a_{i*} \leq 0 \text{ on } [0,1]\}$, $\partial_i = \partial K_i$, and $C_i^- = \{\lambda \in \mathbb{R}^N: \lambda \cdot a_{i*} < 0 \text{ on } [0,1]\}$.

In this section, we will discuss oscillation theorems for the system (4.1), specifically the existence of eigenvalues $\lambda = \lambda^0$ at which simultaneously each of the problems in the system has a non-trivial solution. Often we seek an eigenvalue for which the i^{th} problem has a non-trivial solution with a specified oscillation number n_i, $i = 1,\ldots,N$. The first result does not appear in any of the modern literature:

Theorem 4.2 (Richardson's Necessary Conditions): Suppose that for each $i = 1,\ldots,N$ there exists a sequence of integers $\{n_{i,m}\}_{m=1}^{\infty}$ with $n_{i,m} \to \infty$ as $m \to \infty$ and such that $\bigcap_{i=1}^N S_{n_i}^i \neq \phi$ for all possible choices of $n_i \in \{n_{i,m}\}_{m=1}^{\infty}$, $i = 1,\ldots,N$. Then necessarily for $i = 1,\ldots,N$

$$(4.2) \qquad (\textstyle\bigcap_{j \neq i} \partial_j) \cap (\mathbb{R}^N \, \text{int} \setminus K_i) \neq \phi.$$

Proof: For convenience take $i = 1$. Choose for each m an eigenvalue

$$t_m \lambda^m \in S_{n_{1,m}}^1 \cap (\textstyle\bigcap_{j \geq 2} S_{n_{j,1}}^j) \text{ with } |\lambda^m| = 1,\ t_m \geq 0.$$

We can assume, by taking an appropriate subsequence, that

$\lambda^m \to \lambda^0$ for some $|\lambda^0| = 1$. Since $t_m \geq \rho^1_{n_{1,m}} \to \infty$ as $m \to \infty$ (because of $n_{1,m} \to \infty$ as $m \to \infty$ and Lemma 4.1), necessarily $\lambda^0 \in \bigcap_{j\geq 2} \partial_j$, by Theorem 3.2. If $\lambda^0 \in \text{int } K_1$ then there is an $M \geq 0$ with $\lambda^m \in \text{int } K_1$ for all $m \geq M$. Then $\theta(1;t_m\lambda^m) \leq \theta(1;0)$ for all $m \geq M$, giving a contradiction with $n_{1,m} \to \infty$. ■

We note in particular that if there are integers N_j^* such that $\bigcap_{i=1}^N S^i_{n_i} \neq \phi$ whenever $n_i \geq N_i^*$, $i = 1,\ldots,N$, then the conclusion (4.2) still holds. The reader may wonder if in (4.2) the set $(\mathbb{R}^N \setminus \text{int } K_i)$ can be replaced by $\text{ext } K_i = (\mathbb{R}^N \setminus K_i)$. The answer is no, because of the following example: Let

$$\begin{pmatrix} a_{11}(x_1) & a_{12}(x_1) \\ a_{21}(x_2) & a_{22}(x_2) \end{pmatrix} = \begin{pmatrix} \sin(\pi x_1) & \cos(\pi x_1) \\ 0 & 1 \end{pmatrix}$$

so that $K_1 = \mathscr{R}((-1,0)) =$ (a single ray), $K_2 = \{(\lambda_1,\lambda_2): \lambda_2 \leq 0\}$. We have $K_1 \subset K_2$ so that $\partial_1 \cap \text{ext } K_2 = \phi$. But for the system

$$\begin{cases} y_1''(x_1) + (\lambda_1 \sin \pi x_1 + \lambda_2 \cos \pi x_1) y_1 = 0 \\ y_1(0) = y_1(1) = 0 \\ \\ y_2''(x_1) + \lambda_2 y_2 = 0 \\ y_2(0) = y_2(1) = 0 \end{cases}$$

there exists $\lambda^{n,m} \in S^1_n \cap S^2_m$ for all $n \geq 0$, $m \geq 0$: Choose

$\lambda^{n,m} = (\lambda_1^{n,m}, \lambda_2^{n,m})$ with $\lambda_2^{n,m} = (m\pi)^2$, and find $\lambda_1^{n,m}$ for the problem $y_1'' + (\lambda_1^{n,m} \sin \pi x_1 + (m\pi)^2 \cos \pi x_1) y_1 = 0$, $y_1(0) = y_1(1) = 0$. It is possible to find the $\lambda_1^{n,m}$ by proving a stronger version of Theorem 2.2 with the hypothesis "$r > 0$ except possibly at isolated points" replacing "$r > 0$". It is not too difficult to prove this stronger version of Theorem 2.2.

At present there are several results known which state sufficient conditions for the existence of eigenvalues for the system (4.1) of Sturm-Liouville problems. We will present these results, prove Richardson's theorem on sufficient conditions for the two-parameter problem, show how the other two-parameter results can be deduced as corollaries of Richardson's theorem, and present a new N-parameter result which is not covered by results previously found in the literature.

Define $|A|(x_1,\ldots,x_N) = \det(a_{ij}(x_i))_{\substack{i=1\\j=1}}^{N}$ and the minor subdeterminants

$$a_{ij}^* = a_{ij}^*(x_1,\ldots,x_{i-1},x_{i+1},x_N) = (-1)^{i+j} \det(a_{rs}(x_r))_{\substack{r\neq i\\s\neq j}}$$

known as the "cofactors". Define <u>Right-Definiteness</u> as the condition that $|A|(\cdot)$ be sign-definite (either always positive or always negative). Define <u>Left-Definiteness</u> as the two conditions

(4.3) $\qquad q_i \leq 0,\ \alpha_i < \pi/2 < \beta_i,$ for $i = 1,\ldots,N$

(4.4) There exists $\mu^0 \in \mathbb{R}^N$ such that $\sum_{j=1}^N a_{ij}^* \mu_j^0 > 0$ for $i = 1,\ldots,N$.

Theorem 4.3 (Faierman [5, Chapter 2], Ince [12, pp. 248-251]): If Right-Definiteness holds then for all choices of $n_i \geq 0$ there is a unique $\lambda \in \bigcap_{i=1}^N S_{n_i}^i$.

Theorem 4.4 (Källström and Sleeman [13], Sleeman [19,20]): If Left-Definiteness holds then there exist infinitely-many N-tuples $(n_{1,m},\ldots,n_{N,m})$ of non-negative integers such that $\bigcap_{i=1}^N S_{n_{i,m}}^i \neq \phi$.

Theorem 4.5 (Gregǔs, Neuman, and Arscott [10, p. 434], Sleeman [16,17]): For the two-parameter problem assume that $C_1^- \cap C_2^- \neq \phi$, $C_1^- \cap \text{ext } K_2 \neq \phi$, and $\text{ext } K_1 \cap C_2^- \neq \phi$. Then $S_{n_1}^1 \cap S_{n_2}^2 \neq \phi$ for all $n_1 \geq 0$, $n_2 \geq 0$.

Theorem 4.6: For the two-parameter problem assume that there exists μ^j such that $\mu^j \cdot a_{j*} \leq 0$ with strict inequality somewhere on $[0,1]$, $j = 1,2$, and $\partial_1 \cap \text{ext } K_2 \neq \phi$, $\text{ext } K_1 \cap \partial_2 \neq \phi$. Then there are integers N_j^* such that $S_{n_1}^1 \cap S_{n_2}^2 \neq \phi$ whenever $n_1 \geq N_1^*$, $n_2 \geq N_2^*$.

Theorem 4.6 was stated in Richardson [14, pp. 32-34], and a geometric proof was given there; however, Richardson assumed that the coefficients a_{ij} are analytic and specified the Dirichlet boundary conditions $y_i(0) = y_i(1) = 0$ ($\alpha_i = 0$, $\beta_i = \pi$). These restrictions do affect the generality of the discussion in Richardson [14].

Before proceeding with the proof of this result we need

Lemma 4.7: If there exists μ^1 such that $\mu^1 \cdot a_{1*} \leq 0$ with strict inequality somewhere on $[0,1]$ the S_n^1 is an analytic curve for $n \geq n^*$.

Proof (of the Lemma): After a rotation which sends μ^1 into the positive λ_1-axis we have $a_{11} \leq 0$ with inequality somewhere on $[0,1]$. To prove the result it will suffice to show that there exist functions ϕ_n and sets $\Omega_n \subset \mathbb{R}$ such that $S_n^1 = \{(\phi_n(\lambda_2),\lambda_2): \lambda_2 \in \Omega_n\}$, for all $n \geq n^*$.

Let $\Omega_n = \{\lambda_2^0$: there is a λ_1 with $(\lambda_1,\lambda_2^0) \in S_n^1\}$. Since $\theta(1;\lambda_1,\lambda_2)$ is strictly decreasing in λ_1 for each fixed λ_2 (by comparison theory; see the arguments on page 5), for every $\lambda_2^0 \in \Omega$ there exists a unique λ_1^0 such that $\lambda^0 = (\lambda_1^0,\lambda_2^0) \in S_n^1$. Define $\phi_n(\lambda_2^0) = \lambda_1^0$ in this way. Fix an $n \geq n^*$ and consider the equation $\theta(1;\lambda) - (n\pi+\beta) = 0$. By the Implicit Function Theorem, to show that $\phi_n(\cdot)$ is analytic it will suffice to show that $\partial\theta(1;\lambda^0)/\partial\lambda_1 < 0$ whenever $\lambda^0 \in S_n^1$. It is not difficult to calculate that

$$\partial\theta(1;\lambda^0)/\partial\lambda_1 = \mu(1)^{-1} \int_0^1 \mu_1(x)a_{11}(x)\sin^2\theta(x;\lambda^0)dx$$

where $\mu(x) = \exp(-\int_0^x [-p(s)^{-1} + \lambda^0 \cdot a_{1*}(s) + q(s)]\sin 2\theta(s;\lambda^0)ds)$

is a positive-valued integrating factor. Hence, $\partial\theta(1;\lambda^0)/\partial\lambda_1 \leq 0$. If $\partial\theta(1;\lambda^0)/\partial\lambda_1 = 0$ then necessarily $\theta(x;\lambda^0) \equiv 0(\mathrm{mod}\ \pi)$ for all x where $a_{11}(x) < 0$. Since $a_{11}(x) < 0$ for x in some subinterval of $[0,1]$ there is a contradiction with the fact that

$\theta(x_0) \equiv 0 (\mathrm{mod}\ \pi)$ implies $\theta'(x_0) > 0$. ∎

Somewhat weaker versions of this lemma have appeared in Sleeman [16], Faierman [5], and Richardson [14,15].

<u>Proof</u> (of Theorem 4.6): Since there is a μ^1 such that $\mu^1 \cdot a \leq 0$ with strict inequality somewhere on $[0,1]$, necessarily $K_1 \neq \{0\}$. By Theorems 3.3, 3.4 we can conclude that for $n \geq n_1^*$ the curve S_n^1 is unbounded and asymptotically parallel to the ray or pair of rays which forms ∂_1, where n_1^* is the minimum oscillation number for the first problem. Similarly, the S_n^2 have the same properties with respect to ∂_2 for all $n \geq n_2^*$.

Recall now from the proof of Theorem 3.4 (see page 14) the definition of $\gamma(\cdot)$ in terms of polar coordinates, i.e., $\gamma: \lambda = (r \cos \gamma,\ r \sin \gamma) \mapsto \gamma \in [0, 2\pi)$.

Let $\partial_j = \mathscr{R}_j^- \cup \mathscr{R}_j^+$ = (the union of at most two rays), $\mathscr{R}_j^{\pm} = \mathscr{R}(\lambda_j^{\pm})$ with $|\lambda_j^{\pm}| = 1$, and $\gamma_j^{\pm} = \gamma(\lambda_j^{\pm})$. For convenience assume that $\gamma_j^- \leq \gamma_j^+$, $j = 1,2$. We can assume that $\gamma_1^- \leq \gamma_2^-$, without loss of generality.

We note that if $\gamma_1^- = \gamma_2^-$ then either $K_1 \subset K_2$ or $K_2 \subset K_1$, contradicting the hypotheses $\partial_1 \cap \mathrm{ext}\ K_2 \neq \phi$ and $\mathrm{ext}\ K_1 \cap \partial_2 \neq \phi$. So, without loss of generality, $\gamma_1^- < \gamma_2^-$. Also, $\gamma_1^+ < \gamma_2^+$, because if otherwise then $K_2 \subset K_1$.

Let us define now $\tilde{\gamma}_j^+ = \gamma_j^+$ and $\tilde{\gamma}_j^- = \gamma_j^- + 2\pi$, $j = 1,2$. The hypotheses imply $\tilde{\gamma}_1^+ < \tilde{\gamma}_2^+ < 2\pi \leq \tilde{\gamma}_1^- < \tilde{\gamma}_2^-$. Let

$\eta = \max\{\tilde{\gamma}_2^+ - \tilde{\gamma}_1^+,\ \tilde{\gamma}_1^- - \tilde{\gamma}_2^+,\ \tilde{\gamma}_2^- - \tilde{\gamma}_1^-\}/4.$

The cone K_1 can take either of two forms:

$$K_1 = \{t\lambda:\ t \geq 0,\ \gamma^- \leq \gamma(\lambda) \leq \gamma^+\}$$

or

$$K_1 = \{t\lambda:\ t \geq 0,\ \gamma(\lambda) = \pm\gamma^-\}.$$

The latter can only occur when a_1, a_2 are linearly dependent and $K_1 = \partial_1 = \{\lambda:\ \lambda \cdot a_{1*} = 0$ on $[0,1]\}$. The hypothesis "there exists μ^1 such that $\mu^1 \cdot a_{1*} \leq 0$ with strict inequality somewhere on $[0,1]$" precludes the latter possibility. A similar statement holds for K_2.

Let now $N_j^* = \max\{n \geq 0:\ S_n^j \cap K_j \neq \phi\}$ for $j = 1,2$. Because $\theta_j(1;\cdot)$ is bounded on K_j by $\theta_j(1;0)$, we can conclude that $N_j^* < \infty$.

Let us fix any $n_1 \geq N_1^*$ and $n_2 \geq N_2^*$. For any set $G \subset \mathbb{R}^2$ define $\gamma(G) = \{\gamma(g):\ g \in G\}$. By the definitions of the N_j^* we have that $\gamma(S_{n_j}^j) \cap [\gamma_j^-, \gamma_j^+] = \phi$, so that $\gamma(S_{n_j}^j) \subset [0,\gamma_j^-) \cup (\gamma_j^+, 2\pi)$ for $j = 1,2$. The inclusion is, in fact, equality because $S_{n_j}^j$ is a continuous curve, by Lemma 4.7 proven above, and asymptotically parallel to the ray(s) of ∂_j, by Theorem 3.4.

Let us define now two "wrapped-around-$[0,2\pi)$" forms of $\gamma(\cdot)$ appropriate to $S_{n_1}^1, S_{n_2}^2$, respectively. We let

$$\gamma_j(\lambda) = \begin{cases} \gamma(\lambda) & \lambda \in S^j_{n_j} \quad \text{and} \quad \gamma(\lambda) \in (\gamma_j^+, 2\pi) \\ 2\pi + \gamma(\lambda), & \lambda \in S^j_{n_j} \quad \text{and} \quad \gamma(\lambda) \in [0, \gamma_j^-). \end{cases}$$

With this definition we have $\gamma_j(S^j_{n_j}) = (\tilde{\gamma}_j^+, \tilde{\gamma}_j^-) \subset [0, 4\pi)$.

Recall the choice of η. Because $S^j_{n_j}$ is asymptotically parallel to ∂_j, $j = 1,2$, there is an $R > 0$ sufficiently large that

$$|\lambda| \geq R,\ \lambda \in S^j_{n_j} \quad \text{implies} \quad \gamma_j(\lambda) \in (\tilde{\gamma}_j^+, \gamma_j^+ + \eta) \cup (\tilde{\gamma}_j^- - \eta, \tilde{\gamma}_j^-)$$

and also there exists $\mu_j^\pm$ such that $\mu_j^\pm \in S^j_{n_j}$, $|\mu_j^\pm| = R$, and $\gamma_j(\mu_j^+) \in (\tilde{\gamma}_j^+, \tilde{\gamma}_j^+ + \eta)$, $\gamma_j(\mu_j^-) \in (\tilde{\gamma}_j^- - \eta, \tilde{\gamma}_j^-)$. We have, by the definition of η, the inequalities

$$(4.5) \quad \tilde{\gamma}_1^+ < \gamma_1(\mu_1^+) < \gamma_2(\mu_2^+) < \tilde{\gamma}_2^+ + \eta < \tilde{\gamma}_1^- - \eta < \gamma_1(\mu_1^-) < \gamma_2(\mu_2^-).$$

Define now two curves $\sum_j = \{(\gamma_j(\lambda), |\lambda|) : \lambda \in S^j_{n_j}\}$ in $[0, 4\pi) \times (0, \infty)$. These curves are analytic because $S^1_{n_1}, S^2_{n_2}$ are analytic. This last fact was proven in Lemma 4.7 using the hypothesis "there exists μ^j such that ...". We note that (i) $(\gamma_j(\mu_j^\varepsilon), R) \in \sum_j$, $\varepsilon = \pm$, $j = 1,2$, and (ii) $|\lambda| < R$ whenever $\lambda \in S^j_{n_j}$ and $\gamma_j(\lambda) \in (\tilde{\gamma}_j^+ + \eta, \tilde{\gamma}_j^- - \eta)$. From these two facts we will conclude that there is a point $(\gamma^*, r^*) \in \sum_1 \cap \sum_2$, which implies there is the point $(r^* \cos \gamma^*, r^* \sin \gamma^*) \in S^1_{n_1} \cap S^2_{n_2}$, as desired.

We focus our attention on the interval $I^* = [\tilde{\gamma}_2^+ + \eta, \tilde{\gamma}_1^- - \eta]$, and define $\Lambda_j = \Sigma_j \cap \{\lambda: \gamma_j(\lambda) \in I^*\} \subset \{\lambda: |\lambda| \leq R\}$, by (ii). One may refer to Figure 1 found on page 38. If we assume that $\Sigma_1 \cap \Sigma_2 = \phi$, then either (a) Λ_1 lies above Λ_2, or (b) Λ_2 lies above Λ_1. Here by "Λ_1 lies above Λ_2" we mean that $\lambda^j \in \Lambda_j$ and $\gamma_1(\lambda^1) = \gamma_2(\lambda^2)$ implies $|\lambda^1| > |\lambda^2|$. In case (a), continuing along the curve Σ_1 above Σ_2 we conclude that there is a $\mu \in \Sigma_1$ such that $\gamma_1(\mu) = \gamma_2(\mu_2^+)$ and $|\mu| > R$, giving a contradiction with (ii). In case (b), continuing along the curve Σ_2 above Σ_1 we conclude there is a $\mu \in \Sigma_2$ such that $\gamma_2(\mu) = \gamma_1(\mu_1^-)$ and $|\mu| > R$, giving a contradiction with (ii).

This concludes the proof of the theorem. ■

We note that the statement "$\partial_1 \cap \text{ext } K_2 \neq \phi$ and $\text{ext } K_1 \cap \partial_2 \neq \phi$" is equivalent to "$K_1 \not\subset K_2$ and $K_2 \not\subset K_1$".

It will now be shown that for the two-parameter problem Theorem 4.6 is the strongest result in the literature. Aside from this we note that, upon assuming there exist μ^j such that $\mu^j \cdot a_{j*} \leq 0$ with strict inequality somewhere on $[0,1]$, $j = 1,2$, Theorem 4.6 is "close to sharp" by recalling Theorem 4.2 (Necessary Conditions) and the example which follows it on page 19.

It will be shown in Section 5, that $C^- \neq \phi$ implies $\text{int } K = C^-$, and we will use this fact in the discussion below. We note also that $C^- \neq \phi$ implies the minimum oscillation number $n^* = 0$.

First of all we compare Theorem 4.6 with Theorem 4.3 (Right-Definiteness). It is known that $|A|$ sign-definite implies that $C_1^- \cap C_2^- \neq \phi$ and $C_1^- \cap C_2^+ \neq \phi$, from Sleeman [18, pp. 204-205]. To begin with, $K_1 \not\subset K_2$, because if $K_1 \subset K_2$ we would have $\phi \neq C_2^+ \cap C_1^- \subset C_1^- = \text{int } K_1 \subset K_2 \subset \text{ext } C_2^+$ and $C_2^+ \cap C_1^- \subset C_2^+$. The same argument shows that $K_2 \not\subset K_1$. Theorem 4.6 shows that there exist $N_j^* \geq 0$ such that $S_{n_1}^1 \cap S_{n_2}^2 \neq \phi$ whenever $n_1 \geq N_1^*$, $n_2 \geq N_2^*$. From Theorem 2.2 there exists a $\Lambda^0 \in C_1^- \cap C_2^-$ sufficiently large that $\theta_j(1;0) < \beta_j$ for the translated problems

$$\theta_j'(x_j;\mu) = p_j^{-1}\cos^2\theta_j + (\mu\cdot a + (\Lambda^0\cdot a+q))\sin^2\theta_j, \quad ' = \frac{d}{dx_j},$$

$j = 1,2$. With this we have $N_1^* = N_2^* = 0$. This gives the conclusion of Theorem 4.3, except that no uniqueness is shown.

Second of all we compare Theorem 4.6 with Theorem 4.4 (Left-Definiteness). For $N = 2$ the assumption (4.4) reduces to: there is a $\mu^0 \in \mathbb{R}^2$ such that $0 < \mu_1^0 a_{11}^* + \mu_2^0 a_{12}^* = \mu_1^0 a_{22} - \mu_2^0 a_{21}$ and $0 < \mu_1^0 a_{21}^* + \mu_2^0 a_{22}^* = -\mu_1^0 a_{12} + \mu_2^0 a_{11} = -(\mu_1^0 a_{12} - \mu_2^0 a_{11})$. This implies that $(-\mu_2^0, \mu_1^0) \in C_1^- \cap C_2^+$. As we saw above, this implies that (by Theorem 4.6) there exist N_j^* such that $S_{n_1}^1 \cap S_{n_2}^2 \neq \phi$ whenever $n_1 \geq N_1^*$, $n_2 \geq N_2^*$. Recalling the definition of the N_j^*'s and using the other Left-Definiteness hypothesis (4.3) we conclude that $N_1^* = N_2^* = 0$.

Third of all we compare Theorem 4.6 with Theorem 4.5

(Greguš et al.). The assumption that $\phi \neq C_1^- \cap \text{ext } K_2 =$ $\text{int } K_1 \cap \text{ext } K_2$ implies that $K_1 \not\subset K_2$, and $\phi \neq \text{ext } K_1 \cap C_2^-$ implies $K_2 \not\subset K_1$. This shows the existence of the N_j^* from Theorem 4.6; as before, a translation by a $\Lambda^0 \in C_1^- \cap C_2^-$ shows that $N_1^* = N_2^* = 0$.

Richardson's second article [15] possibly extends his result to the three-parameter problem, and the claim is made there that the method of proof extends to the N-parameter problem , as well. It is difficult, however, to decide if the method is correct for $N \geq 3$ because both of Richardson's articles [14,15] contain inaccuracies. The simplest mistake is the assumption of convexity, both implicitly in the pictures in [14] and explicitly in the arguments in [15]. Also, the technique of cutting the eigensurfaces with planes, for the three-parameter problem, seems cumbersome, if not impossible, to use in the general N-parameter problem. In spite of this, we must admit that it is possible that useful work in Richardson [15] which remains to be discussed.

The result of Greguš et al. can be generalized into an N-parameter result, Theorem 4.8. This result has a Corollary 4.9 for the three-parameter problem, and an example is given to show that the corollary is not vacuous.

<u>Theorem 4.8</u>: For $N \geq 3$ let there be integers $1 \leq k, \ell \leq N$ such that

(4.6) (a) $a_{k,1} > 0$, $a_{\ell,1} > 0$ on $[0,1]$.

(b) $\det(a_{ij}(x_i))_{\substack{i\neq k,\ell \\ j\neq 1,N}}$ is sign-definite

(4.7) $\det(a_{ij}(x_i))_{\substack{i\neq k \\ j\neq N}}$, $\det(a_{ij}(x_i))_{\substack{i\neq \ell \\ j\neq N}}$ are sign-definite

(4.8) $C_k^- \cap \operatorname{ext} K_\ell \neq \phi$ and $\operatorname{ext} K_k \cap C_\ell^- \neq \phi$.

Then $\bigcap_{j=1}^{N} S^j_{n_j} \neq \phi$ for all $n_1 \geq 0,\ldots,n_N \geq 0$.

<u>Proof</u>: Fix any choice of $n_1 \geq 0,\ldots,n_N \geq 0$, and define $S_k = \bigcap_{j\neq k} S^j_{n_j}$, $S_\ell = \bigcap_{j\neq \ell} S^j_{n_j}$. Note that $S_k \cap S_\ell = \bigcap_{j=1}^{N} S^j_{n_j}$. Using hypothesis (4.7) and Theorem 4.3 (Right-Definiteness), for all $\lambda_N \in \mathbb{R}$ there are unique $\phi_{j,k}(\lambda_N)$, $\phi_{j,\ell}(\lambda_N)$, $j = 1,\ldots,N-1$, such that $\Phi_\ell(\lambda_N) \overset{\text{defn}}{=} (\phi_{1,k}(\lambda_N),\ldots,\phi_{N-1,k}(\lambda_N),\lambda_N) \in S_k$ and $\Phi_\ell(\lambda_N) \overset{\text{defn}}{=} (\phi_{1,\ell}(\lambda_N),\ldots,\phi_{N-1,\ell}(\lambda_N),\lambda_N) \in S_\ell$. We show that these functions are analytic: Let $\theta_j(x_j;\lambda)$ be the angle function for the j^{th} problem. We want to solve the (N-1) equations $(\psi_1(\lambda),\ldots,\psi_{k-1}(\lambda),\ \psi_{k+1}(\lambda),\ldots,\psi_N(\lambda)) = 0 \in \mathbb{R}^{N-1}$ where $\psi_j(\lambda) = \theta_j(1;\lambda) - (n_j\pi+\beta_j)$. We know that $\Phi_k(\lambda_N)$ is a solution for all $\lambda_N \in \mathbb{R}^N$, so it only remains to show that these solutions form an analytic curve parametrized by λ_N. This will follow from the Implicit Function Theorem, since the Jacobian is

$$\det(\partial\psi_i(\lambda)/\partial\lambda_j)_{\substack{i\neq k \\ j\neq N}} = \det\left(\mu_i(1)^{-1}\int_0^1 \mu_i(x_i)a_{ij}(x_i)\sin^2\theta_i(x_i;\lambda)dx_i\right)_{\substack{i\neq k \\ j\neq N}}$$

$$= \int_0^1 \cdots \int_0^1 \det(a_{ij}(x_i))_{\substack{i\neq k \\ j\neq N}} \cdot \prod_{i\neq k}((\mu_i(x_i)\sin^2\theta_i(x_i;\lambda)dx_i/\mu_i(1))$$

$\neq 0$ by hypothesis (4.7),

where μ_i are integrating factors, as in the proof of Lemma 4.7. Exactly the same argument applies to $\Phi_\ell(\lambda_N)$.

By hypothesis (4.8) there exists $\bar{\lambda}$, $\underline{\lambda} \in \mathbb{R}^N$ such that

(i) $\bar{\lambda}\cdot a_{k*} < 0$ and $\bar{\lambda}\cdot a_{\ell *}$ has a positive maximum, and

(ii) $\underline{\lambda}\cdot a_{k*}$ has a positive maximum and $\underline{\lambda}\cdot a_{\ell *} < 0$. We can conclude, using the hypothesis (4.6)(a), that there are constants $\bar{t}$, $\underline{t} > 0$ sufficiently large that $\phi_{1,k}(\bar{t}\bar{\lambda}_N) > \bar{t}\bar{\lambda}_1 > \phi_{1,\ell}(\bar{t}\bar{\lambda}_N)$ and $\phi_{1,\ell}(\underline{t}\underline{\lambda}_N) > \underline{t}\underline{\lambda}_1 > \phi_{1,k}(\underline{t}\underline{\lambda}_N)$. By continuity there exists a λ_N^* such that $\phi_{1,k}(\lambda_N^*) = \phi_{1,\ell}(\lambda_N^*)$.

Now consider the (N-2) problems $\psi_i(\lambda) = 0$, $i \neq k$, $i \neq \ell$. Using hypothesis (4.6)(b), Right-Definiteness implies that for all $(\lambda_1^0,\lambda_N^0) \in \mathbb{R}^2$, there is a unique $(\lambda_2^0,\ldots,\lambda_{N-1}^0) \in \mathbb{R}^{N-2}$ such that $\lambda^0 = (\lambda_1^0,\ldots,\lambda_N^0) \in \bigcap_{i\neq k, i\neq \ell} S^i_{n_i} \overset{\text{defn}}{=} S$. In particular, choose $\lambda_1^0 = \phi_{1,k}(\lambda_N^*) = \phi_{1,\ell}(\lambda_N^*)$, $\lambda_N^0 = \lambda_N^*$. Then, since $\Phi_k(\lambda_N^*) \in S_k \subset S$ and $\Phi_\ell(\lambda_N^*) \in S_\ell \subset S$, the uniqueness of λ_i^0, $i \neq 1$, $i \neq N$ implies that $\Phi_k(\lambda_N^*) = \Phi_\ell(\lambda_N^*) \in S$. This proves the existence of an eigenvalue, as desired. ■

<u>Corollary 4.9</u>: For the three-parameter problem, if $a_{11} > 0$, $a_{12} < 0$; $a_{21} > 0$, $a_{22} > 0$; $a_{31} > 0$, $a_{32} > 0$; and $C_2^- \cap \text{ext } K_3 \neq \phi$, $\text{ext } K_2 \cap C_3^- \neq \phi$, then $S^1_{n_1} \cap S^2_{n_2} \cap S^3_{n_3} \neq \phi$ $\forall n_1 \geq 0$, $n_2 \geq 0$, $n_3 \geq 0$.

<u>Proof</u>: Take $k = 2$, $\ell = 3$. Note that (i) $a_{21} > 0$, $a_{31} > 0$, and

$$\det(a_{ij})_{\substack{i\neq 2,3\\ j\neq 1,3}} = a_{12} < 0$$

(ii) $$\det(a_{ij})_{\substack{i\neq 2\\ j\neq 3}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} > 0 \quad \text{and}$$

$$\det(a_{ij})_{\substack{i\neq 3\\ j\neq 3}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0,$$ and apply the previous theorem. ■

Example: The following satisfies the hypotheses of Corollary 4.7:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 2 & -2 & -200 \\ 3-2x_2 & 2 & 2 \\ 2 & 3-2x_3 & 2 \end{pmatrix}$$

has $(0,0,1) \in C_2^- \cap \text{ext } K_3$ and $(-180,100,1) \in \text{ext } K_2 \cap C_3^-$.

In this example $|A|(x_1,x_2,x_3) = 40 - 8(x_2+x_3) + 200\cdot(4-(3-2x_2)(3-2x_3))$ so that $|A|(x_1,1,1) = 624 > 0, |A|(x_1,0,0) = -960 < 0$. So Right-Definiteness is not satisfied. Since $(1,0,0) \in C_1^- \cap C_2^- \cap C_3^-$, by Theorem 5.2 (to be proven in Section 5) the Left-Definiteness condition (4.4) is not satisfied, either. This shows that the corollary is a new result, and so is the theorem from which the corollary is derived.

Section 5: Define cones $K = \{\lambda \in \mathbb{R}^N: \lambda\cdot a(x) \le 0 \text{ for all } x \in [0,1]\}$, $C^- = \{\lambda \in \mathbb{R}^N: \lambda\cdot a(x) < 0 \text{ for all } x \in [0,1]\}$. Since we always assume that not all of the a_j are trivial, necessarily $\lambda\cdot a(x) = 0$ for all $x \in [0,1]$ implies $\lambda \in \partial = \partial K$. In particular, $0 \in \partial$.

There is a technique which helps the geometrical investigation of the cones: Let $a(x) = (a_1(x),\dots,a_N(x)) \in \mathbb{R}^N$ and define the row range $\Gamma = \{a(x): 0 \le x \le 1\}$ = (a closed subset of $\mathbb{R}^N$). This idea, in an abstract setting involving quadratic forms, also appears in Atkinson [1, p. 162].

Define, for any $z \in \mathbb{R}^N$, a set $L^-(z) = \{y \in \mathbb{R}^N: y\cdot z \le 0\}$ where $\cdot$ is the Euclidean inner product. Whenever $z \neq 0$, $L^-(z)$ is a closed half-space; also $L^-(0) = \mathbb{R}^N$. When $z \neq 0$, $\operatorname{int} L^-(z) = \{y: y\cdot z < 0\}$ is an open half-space. We see immediately that $K = \bigcap_{\gamma\in\Gamma} L^-(\gamma)$. We also see that (i) when $0 \in \Gamma$, necessarily $C^- = \phi$, (ii) when $0 \notin \Gamma$, necessarily $C^- = \bigcap_{\gamma\in\Gamma} \operatorname{int} L^-(\gamma)$. This leads to

Theorem 5.1: Either $C^- = \phi$ or $C^- = \operatorname{int} K$.

Proof: When $0 \notin \Gamma$ we must show that

$$\bigcap_{\gamma\in\Gamma} \operatorname{int} L^-(\gamma) = C^- = \operatorname{int} K^- = \operatorname{int} \bigcap_{\gamma\in\Gamma} L^-(\gamma).$$

(i) If $\lambda^0 \in C^-$, let $-d = \max_{\gamma\in\Gamma} \lambda^0\cdot\gamma < 0$, $|\Gamma| = \max_{\gamma\in\Gamma}|\gamma|$ and let B be the open ball about λ^0 of radius $d/(2|\Gamma|)$. Then $B \subset L^-(\gamma)$ for all $\gamma \in \Gamma$, so that $\lambda \in \operatorname{int} K$.

(ii) If $\lambda^0 \in \operatorname{int} K$, there is an open ball B about λ^0 with $B \subset \bigcap_{\gamma\in\Gamma} L^-(\gamma)$. Then $\lambda^0 \in \operatorname{int} L^-(\gamma)$ for all $\gamma \in \Gamma$, so that $\lambda^0 \in C^-$. ■

Recall now condition (4.4), a Left-Definiteness assumption about the coefficients a_{ij} for a system of Sturm-Liouville problems, and the condition of Right-Definiteness, i.e. that the determinant $|A|(x) = |A|(x_1,\ldots,x_N)$ be sign-definite. We have

<u>Theorem 5.2</u>: If $\bigcap_{i=1}^N C_i^- \neq \phi$, then Left-Definiteness (4.4) implies Right-Definiteness.

<u>Proof</u>: For any $N \times N$ matrix $B = (b_{ij})$ let us define the

<u>cofactor</u> $\operatorname{cof} B = (b^*_{ij}) = ((-1)^{i+j}\det(b_{rs})_{\substack{r\neq i\\ s\neq j}})$, the

<u>adjugate</u> $\operatorname{adj} B = (\operatorname{cof} B)^T$, T = transpose, and the

<u>rank</u> $r(B) = \dim \operatorname{Range}(B)$.

Since there is a $\lambda^0 \in \bigcap_{i=1}^N C_i^-$, there is a rotation $\odot$ which maps λ^0 into the positive λ_1-axis, so that the first column of the matrix $A(x)\odot$ consists only of negative entries, where $A(x) = (a_{ij}(x_i))_{i=1,j=1}^{N\ \ N}$. We will need the fact that

$\mathrm{cof}(A(x)\Theta) = \mathrm{cof}(A(x))\Theta$, which can be found in Eves [4, p. 156 #3.10.12 and p. 206 #4.6.7] using the fact that $\Theta^T = \Theta^{-1}$.

Suppose now that contrary to Right-Definiteness there is an $\overline{x}$ such that $|A|(\overline{x}) = 0$, and proceed to show that there is a contradiction. Denote $B = A(\overline{x})\Theta$.

Since $r(B) \leq N - 1$ we can conclude that $r(\mathrm{cof}\ B) \leq 1$, using a fact about adj found in Eves [4, p. 155]. Left-Definiteness (4.4) can be stated as: There exists $\mu^0 \in \mathbb{R}^N$ such that $\mathrm{cof}(A(x))\mu^0 \in (0,\infty) \times \ldots \times (0,\infty)$. This implies that $r(\mathrm{cof}\ B) \geq 1$, and this implies that $r(\mathrm{cof}\ B) = 1$. There must be a vector $0 \neq c \in \mathbb{R}^N$ and constants $\alpha_1,\ldots,\alpha_N \in \mathbb{R}$ such that

$$\mathrm{cof}\ B = \begin{pmatrix} \alpha_1 c^T \\ \vdots \\ \alpha_N c^T \end{pmatrix}.$$ We then have

$$(\mathrm{cof}\ A(\overline{x}))\mu^0 = (\mathrm{cof}\ B)\,\Theta^{-1}\mu^0 = (\Theta c)^T\mu^0 \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} \in (0,\infty) \times \ldots \times (0,\infty).$$

From this we conclude that $\alpha_1,\ldots,\alpha_N$ are all non-zero and of the same sign.

From elementary properties of the determinant, $0 = \sum_{i=1}^N b_{i1} b^*_{ik}$ whenever $k \neq 1$ and $\det B = \sum_{i=1}^N b_{i1} b^*_{i1}$. Since $0 = \det B$, we have for every k

$$0 = \sum_{i=1}^N b_{i1} b^*_{ik} = c_k \sum_{i=1}^N b_{i1}\alpha_i.$$

By the use of the rotation $\ominus$ we have $b_{i1} < 0$ for all i, and all of the α_i's are non-zero and of the same sign. Thus, $c_k = 0$ for all k. This gives $r(B) = 0$, and there is a contradiction. ∎

Acknowledgement

I am grateful to Professor J. K. Hale, my thesis advisor, who has encouraged me and discussed the work with me.

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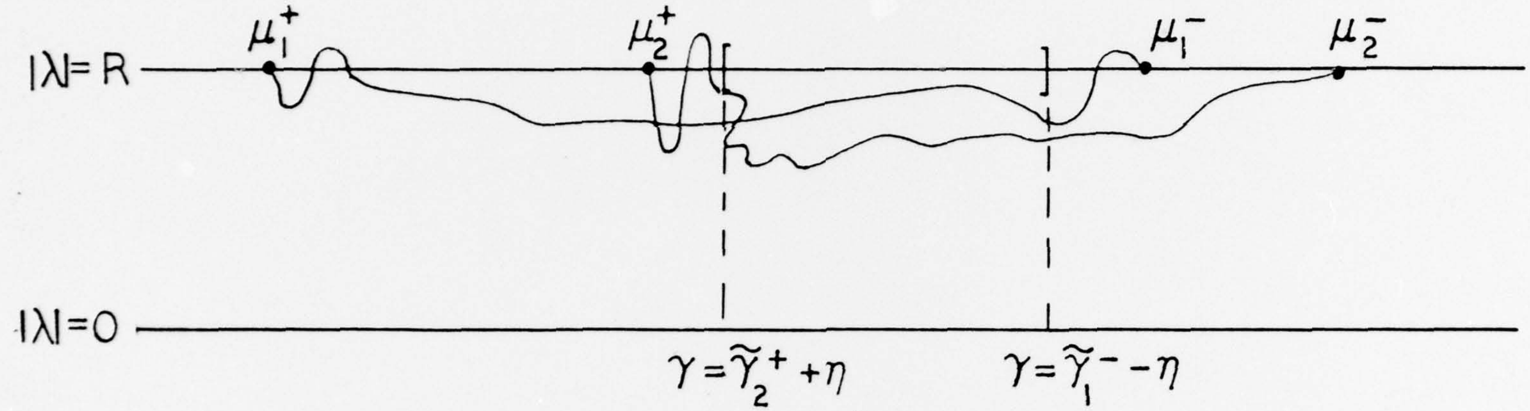

μ_1^+
μ_2^+
μ_1^-
μ_2^-
$|\lambda| = R$
$|\lambda| = 0$
$\gamma = \tilde{\gamma}_2^+ + \eta$
$\gamma = \tilde{\gamma}_1^- - \eta$

FIGURE 1